Teorema (da Wolfram http://mathworld.wolfram.com/Four-DimensionalGeometry.html):

L'ipervolume di una ipersfera in 4D è: $\frac{1}{2}\cdot\pi^2\cdot r^4$

Dimostrazione:

l'equazione di una iperrsfera in 4D è: $x^2+y^2+z^2+w^2=r^2$ da cui $w=\pm\sqrt{r^2-x^2-y^2-z^2}$;

posto $\Omega=(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2<r$ l'ipervolume sarà:

$$hV=\int_{\Omega}\sqrt{r^2-x^2-y^2-z^2}-(-\sqrt{r^2-x^2-y^2-z^2})\,d\Omega \;=\; 2\int_{\Omega}\sqrt{r^2-x^2-y^2-z^2}\,d\Omega$$

facendo un cambiamento di variabili e portandoci in coordinate sferiche:

$$hV=2\int_0^r\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\rho^2\sin(\varphi)\sqrt{r^2-\rho^2\cos^2(\theta)\sin^2(\varphi)-\rho^2\sin^2(\varphi)\sin^2(\theta)-\rho^2\cos^2(\varphi)}\;d\theta\;d\varphi\;d\rho$$

$$hV=2\int_0^r\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\rho^2\sin(\varphi)\sqrt{r^2-\rho^2\sin^2(\varphi)\cdot(\cos^2(\theta)+\sin^2(\theta))-\rho^2\cos^2(\varphi)}\;d\theta\;d\varphi\;d\rho$$

$$hV=2\int_0^r\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\rho^2\sin(\varphi)\sqrt{r^2-\rho^2(\sin^2(\varphi)+\cos^2(\varphi))}\;d\theta\;d\varphi\;d\rho$$

$$hV=2\int_0^r\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\rho^2\sin(\varphi)\sqrt{r^2-\rho^2}\;d\theta\;d\varphi\;d\rho$$

$$hV=4\pi\int_0^r\int_0^{\frac{\pi}{2}}\rho^2\sin(\varphi)\sqrt{r^2-\rho^2}\;d\varphi\;d\rho$$

$$hV=4\pi\;\left[-\cos(\varphi)\right]_0^{\frac{\pi}{2}}\;\int_0^r\rho^2\sqrt{r^2-\rho^2}\;d\rho$$

$$hV=4\pi\;\left[\frac{r^4\arcsin\frac{\rho}{|r|}}{8}-\frac{\rho(r^2-\rho^2)^{\frac{3}{2}}}{4}+\frac{r^2\rho\sqrt{r^2-\rho^2}}{8}\right]_0^R$$

$$hV\;=4\pi\;\left[\frac{\pi r^4}{8}\right]=\frac{1}{2}\pi^2r^4$$

Questa dimostrazione è sostanzialmente uguale alla dimostrazione del volume di una sfera in 4D. Unico punto di interesse è l'utilizzo delle coordinate ipersferiche (http://en.wikipedia.org/wiki/Hypersphere)

Teorema (da Wolfram http://mathworld.wolfram.com/Four-DimensionalGeometry.html):
L'ipervolume di una ipersfera in 5D è: $\frac{8}{15}\cdot\pi^2\cdot r^5$

Dimostrazione:
l'equazione di una ipersfera 5D è: $x^2+y^2+z^2+w^2+v^2=r^2$ da cui $v=\pm\sqrt{r^2-x^2-y^2-z^2-w^2}$;
posto $\Omega=(x,y,z,w)\in\mathbb{R}^4 \mid x^2+y^2+z^2+w^2<R$ l'ipervolume sarà:

$$hV=\int_\Omega \sqrt{r^2-x^2-y^2-z^2-w^2}-(-\sqrt{r^2-x^2-y^2-z^2-w^2})\,d\Omega \;=\; 2\int_\Omega \sqrt{r^2-x^2-y^2-z^2-w^2}\,d\Omega$$

facendo un cambiamento di variabili e portandoci in coordinate ipersferiche:

$$hV=2\int_0^r\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\pi} \rho^3\sin^2(\theta)\sin(\varphi)\sqrt{r^2-\rho^2\cos^2(\theta)-\rho^2\sin^2(\theta)\cos^2(\varphi)-\rho^2\sin^2(\theta)\sin^2(\varphi)\cos^2(\chi)}$$
$$\sqrt{-\rho^2\sin^2(\theta)\sin^2(\varphi)sen^2(\chi)}\;\;d\theta\;d\varphi\;d\chi\;d\rho$$

$$hV=2\int_0^r\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\pi} \rho^3\sin^2(\theta)\sin(\varphi)\sqrt{r^2-\rho^2\cos^2(\theta)-\rho^2\sin^2(\theta)\cos^2(\varphi)}$$
$$\sqrt{-(\rho^2\sin^2(\theta)\sin^2(\varphi))(sen^2(\chi)+\cos^2(\chi))}\;\;d\theta\;d\varphi\;d\chi\;d\rho$$

$$hV=2\int_0^r\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\pi} \rho^3\sin^2(\theta)\sin(\varphi)\sqrt{r^2-\rho^2\cos^2(\theta)-(\rho^2\sin^2(\theta))(\cos^2(\varphi)+\sin^2(\varphi))}\;\;d\theta\;d\varphi\;d\chi\;d\rho$$

$$hV=2\int_0^r\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\pi} \rho^3\sin^2(\theta)\sin(\varphi)\sqrt{r^2-\rho^2(\cos^2(\theta)+\sin^2(\theta))}\;\;d\theta\;d\varphi\;d\chi\;d\rho$$

$$hV=2\int_0^r\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{\pi} \rho^3\sin^2(\theta)\sin(\varphi)\sqrt{r^2-\rho^2}\;\;d\theta\;d\varphi\;d\chi\;d\rho$$

$$hV=4\pi\int_0^r\int_0^{\frac{\pi}{2}}\int_0^{\pi} \rho^3\sin^2(\theta)\sin(\varphi)\sqrt{r^2-\rho^2}\;\;d\theta\;d\varphi\;d\rho$$

$$hV=4\pi\left[-\cos(\varphi)\right]_0^{\frac{\pi}{2}}\int_0^r\int_0^{\pi} \rho^3\sin^2(\theta)\sqrt{r^2-\rho^2}\;\;d\theta\;d\rho$$

$$hV=4\pi\left[\frac{\theta}{2}-\frac{\cos(\theta)\sin(\theta)}{2}\right]_0^{\pi}\int_0^r \rho^3\sqrt{r^2-\rho^2}\;\;d\rho$$

$$hV=2\pi^2\left[\frac{-(r^2-\rho^2)\cdot(2r^2+3\rho^2)}{15}\right]_0^r \;=\; \frac{8}{15}\cdot\pi^2\cdot r^5$$